Tutorial 1

Throughout the tutorial, $R = [0,1]^n$.

Q.1

Prove: If f is an integrable function on a rectangle R, then cf is also integrable on R for any constant c.

Prerequsite:

Recall the integrability condition:

f is integrable on R iff $\forall \varepsilon > 0$, $\exists$ partition P of R such that $\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon$.

P depends on ε

Definition of $\mathcal{U}(f,P)$ & $\mathcal{L}(f,P)$:

$$\mathcal{U}(f,P) = \sum_{C \in P} \left(\sup_{x \in C} f(x) \right) \operatorname{vol}(C)$$

$$\mathcal{L}(f,P) = \sum_{C \in P} \left(\inf_{x \in C} f(x) \right) \operatorname{vol}(C)$$

Be careful: Need to separate the cases $c > 0$ & $c \leq 0$, since if $c < 0$, then $\sup_{x \in R} (cf(x)) = c \inf_{x \in R} f(x)$ (and similar for infimum)

Idea: Write the upper & lower sum of cf in terms of those of f.

Solution:

Case 1: $c>0$

f integrable on R

$\Rightarrow$ Given any $\varepsilon>0$, $\exists$ partion P s.t.

$$\mathcal{U}(f,P)-\mathcal{L}(f,P)<\frac{\varepsilon}{c}$$

$\varepsilon>0$

$\Rightarrow \frac{\varepsilon}{c}>0$

since $c>0$

$$\mathcal{U}(cf,P)=\sum_{C\in P}\sup_{x\in C}(cf(x))\,\mathrm{vol}(C)$$

$$=\sum_{C\in P}c\sup_{x\in C}f(x)\,\mathrm{vol}(C) \qquad (\because c>0)$$

$$=c\sum_{C\in P}\sup_{x\in C}f(x)\,\mathrm{vol}(C)$$

$$=c\,\mathcal{U}(f,P)$$

Similarly, $\mathcal{L}(cf,P)=c\,\mathcal{L}(f,P)$.

$$\Rightarrow \mathcal{U}(cf,P)-\mathcal{L}(cf,P)$$

$$=c(\mathcal{U}(f,P)-\mathcal{L}(f,P))$$

$$<c\cdot\frac{\varepsilon}{c}=\varepsilon$$

Case 2: $c=0$

Then $cf\equiv 0 \Rightarrow \mathcal{U}(cf,P)-\mathcal{L}(cf,P)=0<\varepsilon$.

Case 3: $c<0$

$\varepsilon<0$

$\Rightarrow -\frac{\varepsilon}{c}>0$

Since $c<0$

Given any $\varepsilon>0$,

$\mathcal{U}(f,P)-\mathcal{L}(f,P)<-\frac{\varepsilon}{c}$

$$\mathcal{U}(cf,P) = \sum_{C\in P} \sup_{x\in C}(cf(x))\,\mathrm{vol}(C)$$

$$= \sum_{C\in P} c \inf_{x\in C} f(x)\,\mathrm{vol}(C) \qquad (\because c<0)$$

$$= c\sum_{C\in P} \inf_{x\in C} f(x)\,\mathrm{vol}(C)$$

$$= c\,\mathcal{L}(f,P)$$

Similarly, $\mathcal{L}(cf,P) = c\,\mathcal{U}(f,P)$.

$$\Rightarrow \mathcal{U}(cf,P)-\mathcal{L}(cf,P)$$

$$= -c\left(\mathcal{U}(f,P)-\mathcal{L}(f,P)\right)$$

$$< -c\cdot\left(-\frac{\varepsilon}{c}\right) = \varepsilon$$

Q.2

Let f be a bounded function defined on a rectangle R which is a union of two sub-rectangles R_1 & R_2. Show that that f is integrable on R if and only if f is integrable on both R_1 and R_2.

Prerequisite:

- If P' is a refinement of P, then
  $\mathcal{L}(f, P) \le \mathcal{L}(f, P') \le \mathcal{U}(f, P') \le \mathcal{U}(f, P)$.
  (The lower & upper sum get closer and closer by refinement)

- If $A \subseteq B$, then $\sup_{x \in A} f(x) \le \sup_{x \in B} f(x)$

  $\inf_{x \in A} f(x) \ge \inf_{x \in B} f(x)$

  $\Rightarrow \sup_{x \in A} f(x) - \inf_{x \in A} f(x) \le \sup_{x \in B} f(x) - \inf_{x \in B} f(x)$

Idea: Estimate $\mathcal{U}(f, R) - \mathcal{L}(f, R)$ in terms of $\mathcal{U}(f, R_1) - \mathcal{L}(f, R_1)$ & $\mathcal{U}(f, R_2) - \mathcal{L}(f, R_2)$, and vice versa.

Solution:

($\Rightarrow$) f is integrable of R

$\Rightarrow$ Given any $\varepsilon>0$, $\exists$ partition P of R

$$\mathcal{U}(f,P)-\mathcal{L}(f,P)<\varepsilon$$

Note that $P_1 := \{C\cap R_1 : C\in P\}$ form partitions of

$P_2 := \{C\cap R_2 : C\in P\}$

of R_1 & R_2 respectively.

$$\begin{aligned}
&\mathcal{U}(f,P_1)-\mathcal{L}(f,P_1)\\
&= \sum_{C\in P_1}\left(\sup_{x\in C} f(x)-\inf_{x\in C} f(x)\right)\mathrm{vol}(C)\\
&\overset{*}{\leq} \sum_{C'\in P}\left(\sup_{x\in C'} f(x)-\inf_{x\in C'} f(x)\right)\mathrm{vol}(C')\\
&= \mathcal{U}(f,P)-\mathcal{L}(f,P)\\
&< \varepsilon
\end{aligned}$$

$*$

$\because$ $\forall C\in P_1$,

$C\subseteq C'$ for one $C'\in P$

Also, since

$\sup_{x\in C'} f(x)-\inf_{x\in C'} f(x)\geq 0$,

we can insert rectangles in P that is not in P_1 into P_1 to get the inequality.

$\mathcal{U}(f,P_2)-\mathcal{L}(f,P_2)<\varepsilon$ similarly

$\therefore$ f is integrable on R_1 & R_2.

($\Leftarrow$) f integrable on R_1 & R_2

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ partitions P_1, P_2 of R_1 & R_2 respectively s.t.

$$\begin{cases} U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \\ U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}. \end{cases}$$

We need to construct a partition P of R from P_1 & P_2 such that

$$P_1' = \{ C \cap R_1 : C \in P \} \setminus \{\phi\}$$
$$P_2' = \{ C \cap R_2 : C \in P \} \setminus \{\phi\}$$

are refinements of P_1 & P_2.

Let

$$0 = x_{i,0} < x_{i,1} < \cdots < x_{i,k_i} = 1$$

$$0 = y_{i,0} < y_{i,1} < \cdots < y_{i,h_i} = 1$$

be the grid point defining P_1 & P_2.

Then P can be made by taking all the grid pts above & form a new partition on R.

Then

$$U(f,P)-U(f,P) = \sum_{C\in P} \left(\sup_{x\in C} f(x) - \inf_{x\in C} f(x)\right) \mathrm{vol}(C)$$

∵ $P_1' \cap P_2' \neq \phi$ in general

$$\le \sum_{C\in P_1'} \left(\sup_{x\in C} f(x) - \inf_{x\in C} f(x)\right) \mathrm{vol}(C)$$

$$+ \sum_{C\in P_2'} \left(\sup_{x\in C} f(x) - \inf_{x\in C} f(x)\right) \mathrm{vol}(C)$$

$$= U(f,P_1') - L(f,P_1')$$

$$+ U(f,P_2') - L(f,P_2')$$

∵ P_1', P_2' are refinements of P_1, P_2 respectively

$$\le U(f,P_1) - L(f,P_1)$$

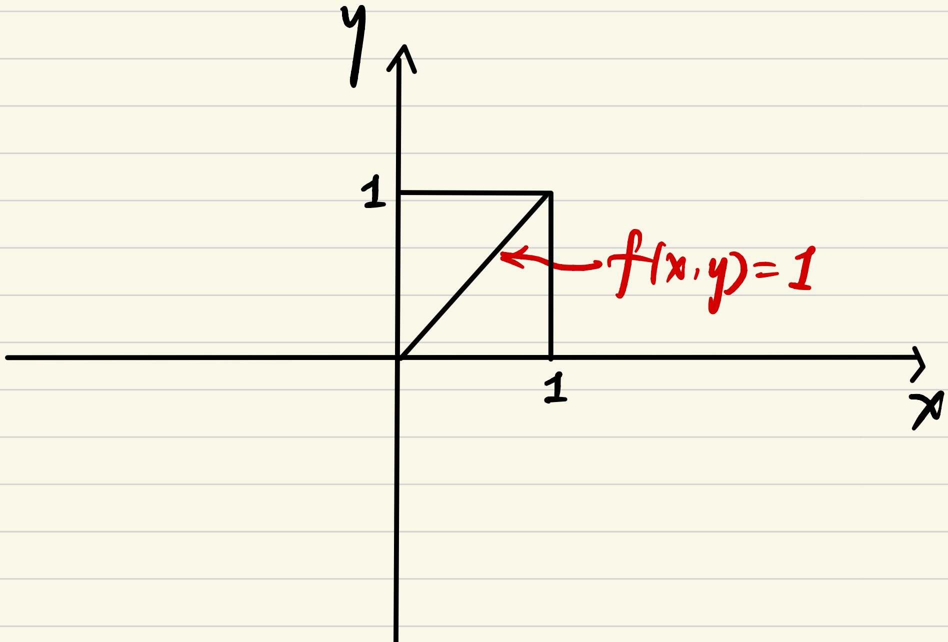
$$+ U(f,P_2) - L(f,P_2)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Q.3

Show, by definition that the function $f(x,y)=1$ when $x=y$, and $f(x,y)=0$ otherwise is integrable on $R=[0,1]\times[0,1]$ and find the integral of f on R.

Prerequisite:



Intuitively, the graph $z=f(x,y)$ has zero volume under the graph, so we should expect that $\int_R f=0$.

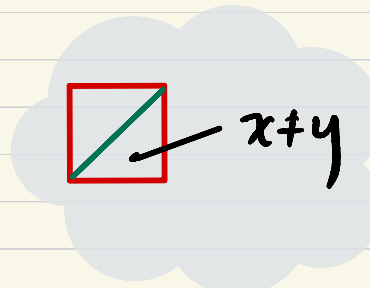
Solution:

Let $\varepsilon > 0$. Let P_n be the partition of R defined as

$$P_n = \{C_{i,j} := [\tfrac{i-1}{n}, \tfrac{i}{n}] \times [\tfrac{j-1}{n}, \tfrac{j}{n}] : 1 \le i,j \le n\}$$

Note that $\mathcal{L}(f, P_n) = 0 \;\; \forall n$ since $\inf_{x \in C_{i,j}} f(x) = 0$

$\forall\, i,j.\ (\forall C_{i,j},\ \exists\, (x,y) \in C_{i,j}$ s.t. $x \neq y)$

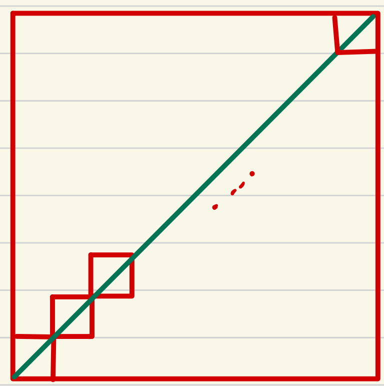


$U(f, P_n)$

$$= \sum_{1 \le i,j \le n} \sup_{x \in C_{i,j}} f(x)\, \mathrm{vol}(C_{i,j})$$

$$= 1 \cdot \frac{1}{n^2} \cdot n$$

$$= \frac{1}{n}$$



Only the diagonal squares intersect with the line $x=y$ $\Rightarrow \sup_{x \in C_{i,j}} f(x) = \begin{cases} 1 & \text{if } C_{i,j} \text{ diagonal} \\ 0 & \text{otherwise.} \end{cases}$ (n squares)

$\therefore$ If we pick N such that $\frac{1}{N} < \varepsilon$, we have

$$U(f, P_N) - \mathcal{L}(f, P_N) = \frac{1}{N} - 0 < \varepsilon$$

$\therefore f$ is integrable on R.

$\therefore \int_R f = \sup_P \mathcal{L}(f, P) = 0$ since $\mathcal{L}(f, P) = 0 \;\forall$ partition P.